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Metric Projection Bound and the Lipschitz Constant of the Radial Retraction

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Let P_M denote the metric projection on a proximinal subspace M of a real normed linear space X. Let $\|P_M\| = \text{Sup}\{\|y\|: y \in P_M(x), \|x\| \leqslant 1\}$. It is shown that the Lipschitz constant for the radial retraction of the unit ball of X is equal to the metric projection bound, which is defined to be $\text{MPB}(X) = \text{Sup}\{\|P_M\| : M$ proximinal subspace of $X\}$. A formula for $\text{MPB}(l_p^2)$, $1 < p < \infty$, is derived in the end.

1. INTRODUCTION

Let X be a real normed linear space, and M a nontrivial closed proper subspace of X. The (possibly empty) set of best approximations to x from M is defined by

$$P_M(x) = \{y \in M : \|x - y\| = d(x, M)\},$$

where $d(x, M) = \inf\{\|x - y\| : y \in M\}$. The subspace M is called proximinal if $P_M(x)$ contains at least one point for every $x \in X$. The mapping $P_M : X \to 2^X$ is called the metric projection onto M. If M is proximinal, the norm of P_M is defined by $\|P_M\| = \text{Sup}\{\|y\| : y \in P_M(x), \|x\| \leqslant 1\}$. It is easily seen that $1 \leqslant \|P_M\| \leqslant 2$ for every proximinal subspace M of X. The metric projection bound of X written as $\text{MPB}(X)$ is defined to be $\text{MPB}(X) = \text{Sup}\{\|P_M\| : M$ proximinal subspace of $X\}$. If X is a Hilbert space then $\text{MPB}(X) = 1$. In general $1 \leqslant \text{MPB}(X) \leqslant 2$. Deutsch and Lambert [3] have constructed a Chebyshev subspace in $C[0, 1]$ whose metric projection is linear and has norm two. Smith [5] has recently characterized uniformly nonsquare Banach spaces as precisely those which have $\text{MPB}(X)$ less than two. Recall that a Banach space X is uniformly nonsquare if there is a positive number δ such that there do not exist elements x and y of the unit ball for which $\|(x + y)/2\| > 1 - \delta$ and $\|(x - y)/2\| > 1 - \delta$. Earlier in [6]

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Thele had proved a similar characterization of uniformly nonsquare Banach spaces as those spaces whose Lipschitz constant is less than two. Let us recall that the Lipschitz constant $k(X)$ of X is the infimum of all numbers k for which $\|Tx - Ty\| \leqslant k\,\|x - y\|$ for all $x, y \in X$. Here T is the radial retraction on the unit ball, defined by

$$\begin{aligned} Tx &= x, && \text{if } \|x\| \leqslant 1, \\ &= \frac{x}{\|x\|}, && \text{if } \|x\| \geqslant 1. \end{aligned}$$

It is also known [2] that $k(X) = 1$ if and only if the Birkhoff–James (B–J) orthogonality (defined below) is symmetric. In [5] it is shown that $\mathrm{MPB}(X) = 1$ if and only if the B–J orthogonality is symmetric. In this paper we show that, in fact, the metric projection bound and the Lipschitz constant are equal for any normed space X. Thus it is not mere chance that the main results of Thele [6] and Smith [5] about uniformly nonsquare spaces, and about symmetry of orthogonality mentioned above look so similar. In the end we obtain a formula for the Lipschitz constant of l_p^2 which leads us to some interesting questions regarding certain inequalities involving l_p norms.

The tool for proving our main result quickly is the B–J orthogonality. The vector x is said to be orthogonal to y, written as $x \perp y$, if $\|x + \alpha y\| \geqslant \|x\|$ for all real numbers α. It is easily seen that $x \perp y$ if and only if there is an $f \in S(X^*)$, the unit ball of the conjugate space X^*, such that $f(x) = \|x\|$ and $f(y) = 0$. It is also known that for each pair of vectors x and y there exist numbers α and β such that $x + \alpha y \perp x$ and $x \perp x + \beta y$. The orthogonality is called symmetric if $x \perp y$ implies $y \perp x$. It is well known that the orthogonality is symmetric in a space of dimension greater than two only if it is an inner product space. For details one can see James [4] and Day [1].

2. The Main Result

THEOREM 1. *For any normed space X, the metric projection bound* $\mathrm{MPB}(X)$ *and the Lipschitz constant $k(X)$ are equal.*

To prove the theorem we first prove

LEMMA 1. *We have* $\mathrm{MPB}(X) = \mathrm{Sup}\{\|P_y\| \colon y \in X\}$. (Here $P_y \equiv P_{[y]}$).

Proof. Clearly $\mathrm{Sup}\{\|P_y\| \colon y \in X\} \leqslant \mathrm{MPB}(X) = m$ (say). Let $\varepsilon > 0$; choose M a proximinal subspace such that $\|P_M\| > m - \varepsilon$. Then there exist $x \in X$ and $y \in P_M(x)$ such that $\|y\| > m - \varepsilon$. Also $\|x - y\| \leqslant \|x - z\|$ for every $z \in M$; therefore, $\|x-y\| \leqslant \|x-ty\|$ for every $t \in R$ and hence $y \in P_y(x)$. Then $\|P_y\| \geqslant \|y\| > m-\varepsilon$, which proves the lemma.

Proof of Theorem. Let $x \perp y$. Then $y \in P_y(x+y)$ and therefore $\|y\|/\|x+y\| \leqslant \|P_y\| \leqslant \mathrm{MPB}(X)$. Thus $\mathrm{Sup}\{\|y\|/\|x+y\|: x \perp y\} \leqslant \mathrm{MPB}(X) = m$. On the other hand if $\varepsilon > 0$, choose $y \in X$ such that $\|P_y\| > m-\varepsilon$. Let z be such that $m-\varepsilon \leqslant \|b\|/\|z\|$ for some $b \in P_y(z)$; then $b \in ty$ for some $t \in R$ and hence $b \in P_b(z)$, giving that $z-b \perp b$. Thus

$$m-\varepsilon \leqslant \mathop{\mathrm{Sup}}_{x \perp y} (\|y\|/\|x+y\|),$$

hence

$$m = \mathop{\mathrm{Sup}}_{x \perp y} (\|y\|/\|x+y\|).$$

Using the result of Thele [6, Theorem 1] that

$$\begin{aligned} k(X) &= \mathrm{Sup}\{\|y\|/\|\alpha x - y\| : y \neq 0, x \perp y, \alpha \in R\} \\ &= \mathrm{Sup}\{\|y\|/\|x+y\| : x \perp y\} \end{aligned}$$

We get the result of the theorem.

3. Metric Projection Bound for l_p^2

It is easily seen that $\mathrm{MPB}(l_p^2) = 2$ if $p=1$ or ∞. If $p \neq 1$ or ∞, then l_p is smooth. The normalized duality map $J: l_p \to l_q = l_p^*$ is given by $J(0)=0$ and $J(x) = \sum |x_i|^{p-1} \operatorname{sgn} x_i/\|x\|^{p-2}$ for $0 \neq x = (x_i)$. If $0 \neq x$ and $y=(y_i)$, then $x \perp y$ if and only if $(J(x), y) = \sum |x_i|^{p-1} y_i \operatorname{sgn} x_i/\|x\|^{p-2} = 0$.

In what follows we will use the notation $\|x\|_r = (\sum |x_i|^r)^{1/r}$ even when $0 < r < 1$.

THEOREM 2. *For* $1 < p < \infty$,

$$\mathrm{MPB}(l_p^2) = k(l_p^2) = \mathop{\mathrm{Sup}}_{x \in l_p^2} [\|x\|_{p(p-1)}^{p-1} \|x\|_q/\|x\|_p^p].$$

Proof. By Theorem 1, $\mathrm{MPB}(X) = k(X) = \mathrm{Sup}_{x \perp y}(\|y\|/\|x+y\|)$. It is easily seen that

$$1/k(X) = \mathop{\mathrm{Inf}}_{\substack{\alpha \in R \\ x \perp y \\ \|y\|=1}} \|\alpha x - y\| = \mathop{\mathrm{Inf}}_{\substack{\|y\|=1 \\ x \perp y \\ \alpha x - y \perp x}} \|\alpha x - y\|$$

This means that for l_p^2 we have to find the minimum value of $\|\alpha x - y\|_p$ under the constraints that

$$|y_1|^p + |y_2|^p = 1, \quad (1)$$

$$y_1 |x_1|^{p-1} \operatorname{sgn} x_1 + y_2 |x_2|^{p-1} \operatorname{sgn} x_2 = 0, \quad (2)$$

$$x_1 |\alpha x_1 - y_1|^{p-1} \operatorname{sign}(\alpha x_1 - y_1) + x_2 |\alpha x_2 - y_2|^{p-1} \operatorname{sgn}(\alpha x_2 - y_2) = 0. \quad (3)$$

We will assume that $0 < x_1 < 1$, $0 < x_2 < 1$, $y_1 < 0$, $y_2 > 0$, and $\alpha > 0$. The other cases are similarly dealt with. Now,

$$\begin{aligned}\|\alpha x - y\|_p^p &= \|\alpha x - y\|^{p-2}(J_{(\alpha x - y)}, \alpha x - y)\\ &= \|\alpha x - y\|^{p-2}(J_{(\alpha x - y)}, -y)\\ &= -(y_1 |\alpha x_1 - y_1|^{p-1} \operatorname{sgn}(\alpha x_1 - y_1)\\ &\quad + y_2 |\alpha x_2 - y_2|^{p-1} \operatorname{sgn}(\alpha x_2 - y_2)).\end{aligned}$$

Putting the value of $|\alpha x_2 - y_2|^{p-1} \operatorname{sgn}(\alpha x_2 - y_2)$ from (3) we get

$$\|\alpha x - y\|^p = ((y_2 x_1 - y_1 x_2)/x_2) |\alpha x_1 - y_1|^{p-1} \operatorname{sgn}(\alpha x_1 - y_1)$$

and

$$x_1(\alpha x_1 - y_1)^{p-1} = x_2(y_2 - \alpha x_2)^{p-1},$$

which yields

$$\alpha = (y_1 x_1^{1/(p-1)} + y_2 x_2^{1/(p-1)})/(x_1^{p/(p-1)} + x_2^{p/(p-1)}).$$

We can rewrite condition (2) as $-y_1 x_1^{p-1} = y_2 x_2^{p-1}$, and combining this with (1) we finally get

$$x_1 y_2 - y_1 x_2 = (x_1^p + x_2^p)/(x_1^{p(p-1)} + x_2^{p(p-1)})^{1/p},$$

and

$$\begin{aligned}\|\alpha x - y\|_p &= (|x_1|^p + |x_2|^p)/(|x_1|^q + |x_2|^q)^{1/q}(|x_1|^{p(p-1)} + |x_2|^{p(p-1)})^{1/p}\\ &= \|x\|_p^p/\|x\|_q \, \|x\|_{p(p-1)}^{(p-1)}.\end{aligned}$$

From this the result follows.

Remark 1. Theorem 2 raises the following questions about norm inequalities in l_p spaces:

(i) Is $k(l_p^n) = \operatorname{Sup}_{x \in l_p^n}(\|x\|_{p(p-1)}^{(p-1)} \|x\|_q/\|x\|_p^p)$?

If the answer is yes, then we shall have

$$1 \leqslant \|x\|_{p(p-1)}^{p-1} \|x\|_q / \|x\|_p^p \leqslant 2.$$

(ii) Is $1 \leqslant \|x\|_{p(p-1)}^{(p-1)} \|x\|_q / \|x\|_p^p \leqslant 2$ for $x \in l_p$ or l_p^n?

The first inequality in (ii) follows from the convexity of the function $f(r) = \log \|x\|_r^r$ for $0 < r < \infty$.

Remark 2. We can see that $k(l_p^2)$ is the maximum value of $((1 + t^{p(p-1)})^{1/p}(1 + t^q)^{1/q})/1 + t^p$ on the interval $0 \leqslant t \leqslant 1$.

For $p = 3$ and 4 we have been able to obtain the exact values of $k(l_3^2)$ and $k(l_4^2)$ which are $\frac{1}{3}(17 + 7\sqrt{7})^{1/3}$ and $(1 + \frac{2}{3}\sqrt{3})^{1/4}$, respectively.

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